Effective discretization for regularized algebraic reconstruction techniques

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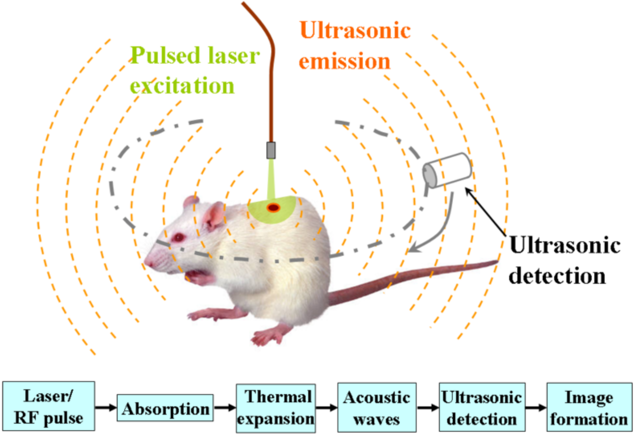
Outline

Photoacoustic Imaging



Source:http://en.wikipedia.org/wiki/Photoacoustic_imaging_in_biomedicine

Wave Equation

- Cauchy problem

$$\begin{aligned} \partial_t^2 p(\mathbf{x},t) &= \Delta p(\mathbf{x},t) \quad \text{for} \quad (\mathbf{x},t) \in \mathbb{R}^3 \times (0,\infty)\,, \\ p(\mathbf{x},0) &= f(\mathbf{x}) \qquad\qquad\qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^3\,, \\ \partial_t p(\mathbf{x},0) &= 0 \qquad\qquad\qquad\quad\;\; \text{for} \quad \mathbf{x} \in \mathbb{R}^3\,. \end{aligned}$$

- Solution

$$p(\mathbf{y},t) = \partial_t \left(t \left(\mathscr{M} f \right) (\mathbf{y},t) \right)$$

Spherical mean value operator for $d \geq 2$

$$\mathscr{M} f(\mathbf{y},t) := \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(\mathbf{y} + t\xi)\, \mathrm{d}\sigma(\xi), \quad (\mathbf{y},t) \in \mathbb{R}^d \times (0,\infty)$$

Inverse Problem

- Measurements

$$\mathbf{g} \approx \mathscr{M} f(\mathbf{y}, t), \quad \mathbf{y} \in Y, \quad t \in T$$

  for given detector positions $Y = \{\mathbf{y}_1 \ldots \mathbf{y}_{M_1}\} \subset \mathbb{R}^3$ and time points $T = \{t_1, \ldots, t_{M_2}\} \subset [0, \infty)$
- Task: Find f such that

$$\mathscr{M} f = g$$

Tikhonov Regularization

$$f^{+} = \arg\min_{f} \| \mathscr{M} f - g \|_2^2 + \alpha \Phi(f)$$

Tikhonov Regularization

$$f^+ = \arg\min_f \| \mathscr{M} f - g \|_2^2 + \alpha \| f \|_1$$

- Assumption: f^+ has sparse representation in $\mathscr{B} = \{\psi_i\}_{i\in\mathbb{N}}$:

$$f^+ = \sum_{i\in\mathbb{N}} c_i \psi_i \,.$$

Tikhonov Regularization

$$f^{+} = \arg\min_{\mathbf{c}} \|K\mathbf{c} - g\|_2^2 + \alpha \|\mathbf{c}\|_1$$

- Assumption: f^{+} has sparse representation in $\mathscr{B} = \{\psi_i\}_{i\in\mathbb{N}}$:

$$f^{+} = \sum_{i\in\mathbb{N}} c_i \psi_i \,.$$

- Synthesis operator

$$S\left((c_i)_{i\in\mathbb{N}}\right) = \sum_{i\in\mathbb{N}} c_i \psi_i \,.$$

- $K = \mathscr{M} \circ S$

Outline

Property of $\mathscr{M}$

Lemma

For $e_\xi(\mathbf{x}) = e^{2\pi i \xi \cdot \mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $\mathbf{y} \in \mathbb{R}^d$ *and* $t \geq 0$ *it holds that*

$$\left(\mathscr{M} e_\xi\right)(\mathbf{y}, t) = \begin{cases} e_\xi(\mathbf{y}) \dfrac{\Gamma\left(\frac{d}{2}\right) J_{\frac{d-2}{2}}(2\pi|\xi|_2 t)}{(\pi|\xi|_2 t)^{\frac{d-2}{2}}}, & \xi \in \mathbb{R}^d \setminus \{\mathbf{0}\}, \\ 1, & \xi = \mathbf{0}. \end{cases}$$

Approach

Idea

$$f(\mathbf{x}) \approx \sum_{\xi \in \Xi} \hat{f}_\xi \mathrm{e}^{2\pi\mathrm{i}\xi\cdot\mathbf{x}} \quad \text{with } \Xi \subset \mathbb{R}^d \text{ finite, } \hat{f}_\xi \in \mathbb{C}$$

$$\Rightarrow \quad \mathscr{M} f(\mathbf{y}, t) = \sum_{\xi \in \Xi} \underbrace{\hat{f}_\xi \frac{\Gamma\left(\frac{d}{2}\right) J_{\frac{d-2}{2}}\left(2\pi \left|\xi\right|_2 t\right)}{\left(\pi \left|\xi\right|_2 t\right)^{\frac{d-2}{2}}}}_{h_{\xi,t}} \mathrm{e}^{2\pi\mathrm{i}\xi\cdot\mathbf{y}}$$

$$= \sum_{\xi \in \Xi} h_{\xi,t} \mathrm{e}^{2\pi\mathrm{i}\xi\cdot\mathbf{y}}$$

Torsten Görner, Ralf Hielscher, and Stefan Kunis (2012). "Efficient and accurate computation of spherical mean values at scattered center points." In: *Inverse Probl. Imaging* 6.4, pp. 645–661

Fast Computation Using Fourier Methods

- For each $\xi \in J_N^d$ compute with FFT

$$\hat{f}_\xi = \frac{1}{N^d} \sum_{\xi \in X_N^d} f(x) \mathrm{e}^{2\pi \mathrm{i} \xi \cdot \mathbf{x}}$$

- Computation for each $t \in \{t_1, \dots t_{M_2}\}$ and $\xi \in J_N^d \setminus 0$ of

$$h_{\xi,t} = \hat{f}_\xi \frac{\Gamma\left(\frac{d}{2}\right) J_{\frac{d-2}{2}}\left(2\pi \left|\xi\right|_2 t\right)}{\left(\pi \left|\xi\right|_2 t\right)^{\frac{d-2}{2}}}$$

- For each $\mathbf{y} \in \{\mathbf{y}_1, \dots, \mathbf{y}_{M_1}\}$ compute via NFFT

$$\mathscr{M} f(\mathbf{y}, t) = \sum_{\xi \in \Xi} h_{\xi,t} \mathrm{e}^{2\pi \mathrm{i} \xi \cdot \mathbf{y}}$$

- Complexity for $d = 3$, $M_1 = \mathscr{O}\left(N^2\right)$, $M_2 = \mathscr{O}(N)$:

$$\mathscr{O}\left(N^4 \log N\right)$$

Jens Keiner, Stefan Kunis, and Daniel Potts. *NFFT 3.0, C subroutine library.* http://www.tu-chemnitz.de/~potts/nfft

Outline

Shearlets

Definition (Shearlets)

$$\begin{aligned} \psi_{a,s,\mathbf{t}} : \mathbb{R}^2 &\to \mathbb{R} \\ \mathbf{x} &\mapsto a^{-\frac{3}{4}} \psi\left(\mathbf{A}_a^{-1} \mathbf{S}_s^{-1} (\mathbf{x} - \mathbf{t})\right) \end{aligned}$$

- Translation $\mathbf{t} \in \mathbb{R}^2$
- Dilation $\mathbf{A}_a := \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$, $a > 0$
- Shearing $\mathbf{S}_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, $s \in \mathbb{R}$

Sören Häuser (Preprint). "Fast Finite Shearlet Transform: a tutorial." In: *Preprint University of Kaiserslautern*

Mother Shearlet Ψ

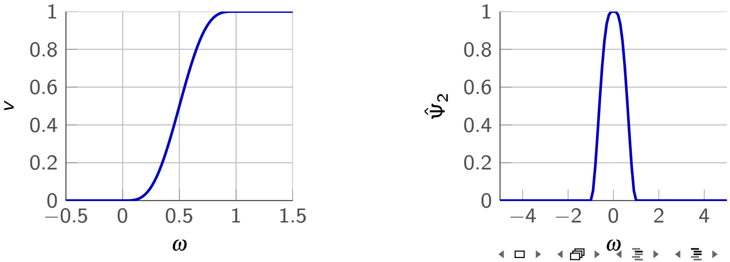
$$
\begin{aligned}
\psi : \mathbb{R}^2 &\rightarrow \mathbb{R}, \\
\hat{\psi}(\omega_1, \omega_2) &:= \hat{\psi}_1(\omega_1)\hat{\psi}_2\left(\frac{\omega_2}{\omega_1}\right)
\end{aligned}
$$

Construction of Ψ_2

- Auxilary function $v : \mathbb{R} \to \mathbb{R}$,

$$v(x) := \begin{cases} 0 & \text{for } x < 0 \\ 35x^4 - 84x^5 + 70x^6 - 20x^7 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

- $\hat{\psi}_2(\omega) := \begin{cases} \sqrt{v(1+\omega)} & \text{for } \omega \le 0 \\ \sqrt{v(1-\omega)} & \text{for } \omega > 0 \end{cases}$

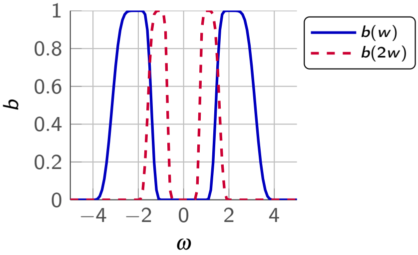


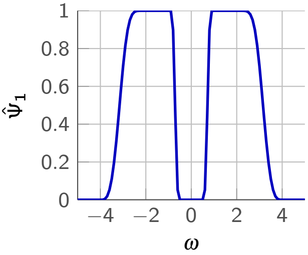
Construction of Ψ_1

- Auxiliary function $b : \mathbb{R} \to \mathbb{R}$,

$$b(\omega) := \begin{cases} \sin\left(\frac{\pi}{2} v(|\omega| - 1)\right) & \text{for } 1 \le |\omega| \le 2 \\ \cos\left(\frac{\pi}{2} v\left(\frac{1}{2}|\omega| - 1\right)\right) & \text{for } 2 < |\omega| \le 4 \\ 0 & \text{otherwise} \end{cases}$$

- $\hat{\Psi}_1(\omega) := \sqrt{b^2(2\omega) + b^2(\omega)}$

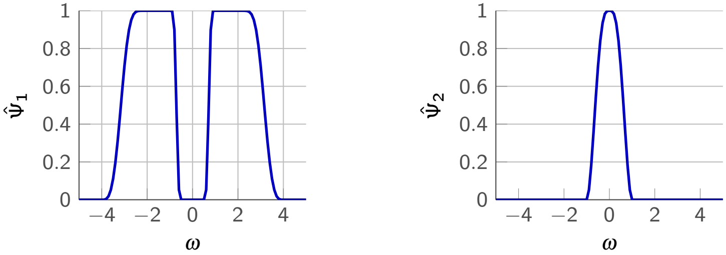




Mother Shearlet Ψ

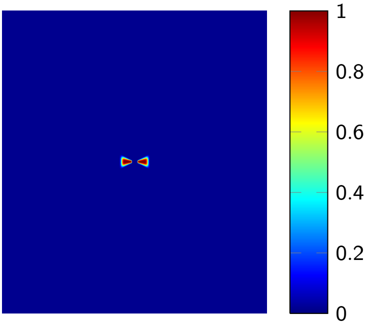
$$\psi : \mathbb{R}^2 \to \mathbb{R}, \quad \hat{\psi}(\omega_1, \omega_2) := \hat{\psi}_1(\omega_1)\hat{\psi}_2\left(\frac{\omega_2}{\omega_1}\right)$$

$$\hat{\psi}_1(\omega) := \sqrt{b^2(2\omega) + b^2(\omega)} \quad \hat{\psi}_2(\omega) := \begin{cases} \sqrt{v(1+\omega)} & \text{for } \omega \le 0 \\ \sqrt{v(1-\omega)} & \text{for } \omega > 0 \end{cases}$$

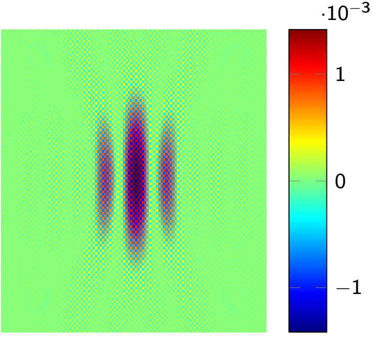


Examples

Shearlet $\hat{\Psi}$ for scale $j = 1$, $a_j = 1/4$ and $s_j = 0$

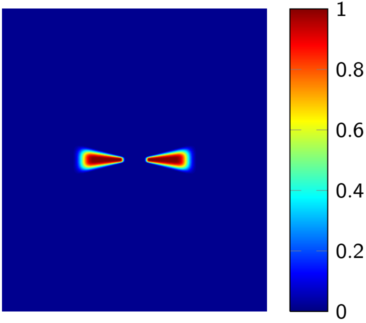


Shearlet Ψ for scale $j = 1$, $a_j = 1/4$ and $s_j = 0$

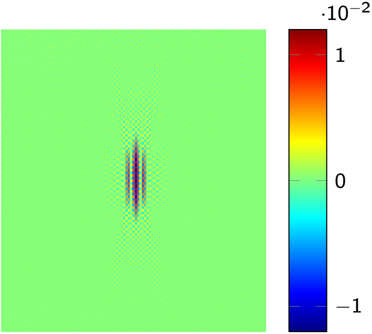


Examples

Shearlet $\hat{\Psi}$ for scale $j = 2$, $a_j = 1/16$ and $s_j = 0$

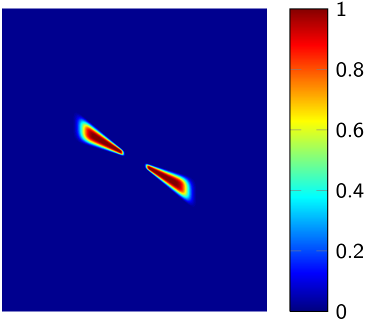


Shearlet Ψ for scale $j = 2$, $a_j = 1/16$ and $s_j = 0$

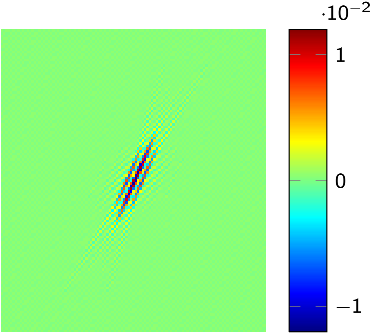


Examples

Shearlet $\hat{\Psi}$ for scale $j = 2$, $a_j = 1/16$ and $s_j = -0.5$

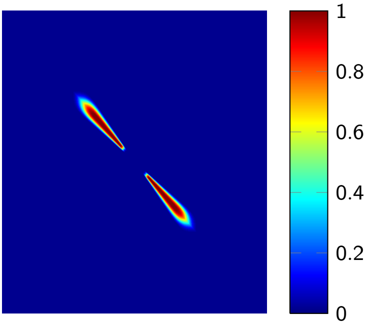


Shearlet Ψ for scale $j = 2$, $a_j = 1/16$ and $s_j = -0.5$

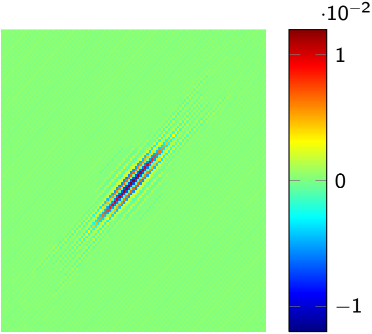


Examples

Shearlet $\hat{\Psi}$ for scale $j = 2$, $a_j = 1/16$ and $s_j = -1$



Shearlet Ψ for scale $j = 2$, $a_j = 1/16$ and $s_j = -1$



Shearlet Transform

Definition (Shearlet Transform)

For $f \in L^2(\mathbb{R}^2)$, $\mathbf{t} \in \mathbb{R}$, $a \in (0,1]$ and $|s| \leq 1$ we define

$$S(f)(a,s,\mathbf{t}) := \langle f \,|\, \psi_{a,s,\mathbf{t}} \rangle$$

$$\begin{aligned} S(f)(a,s,\mathbf{t}) &:= \langle f \,|\, \psi_{a,s,\mathbf{t}} \rangle \\ &= \left\langle \hat{f} \,|\, \hat{\psi}_{a,s,\mathbf{t}} \right\rangle \\ &= a^{3/4} \mathscr{F}^{-1} \left(\hat{f}\,(\omega)\, \hat{\psi}_1\,(\alpha\omega_1)\, \hat{\psi}_2 \left(a^{-\frac{1}{2}} \left[\frac{\omega_2}{\omega_1} + s \right] \right) \right) (t) \end{aligned}$$

Outline

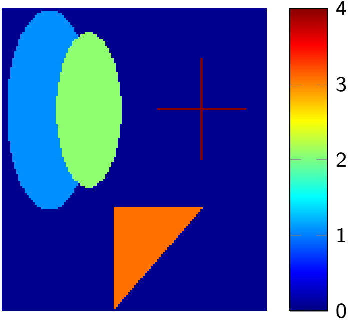
Tikhonov Regularization With l_1-penality

- $$c^+ = \arg\min_c \|Kc - g\|_2^2 + \alpha \|c\|_1$$
- with $K := \mathscr{M} \circ S^*$
- Minimization with help of the regularized feature sign search - algorithm (RFSS)

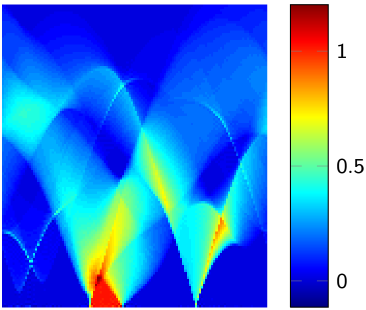
Bangti Jin, Dirk A. Lorenz, and Stefan Schiffler (2009). "Elastic-Net Regularization: Error estimates and Active Set Methods." In: *Inverse Problems* 25.11, 115022 (26pp)

Example

Original image

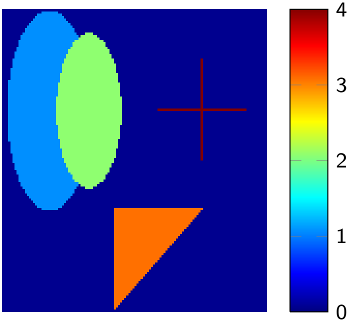


Data g

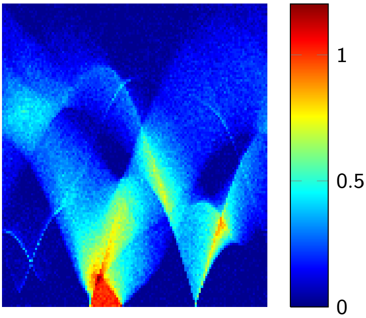


Example

Original image

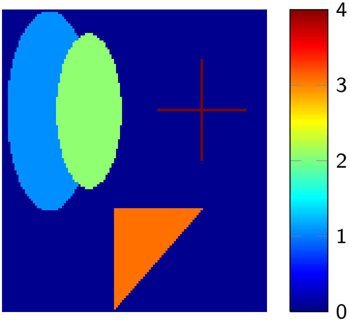


Noisy data g^δ (5 percent)

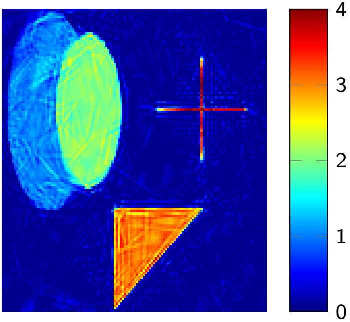


Example

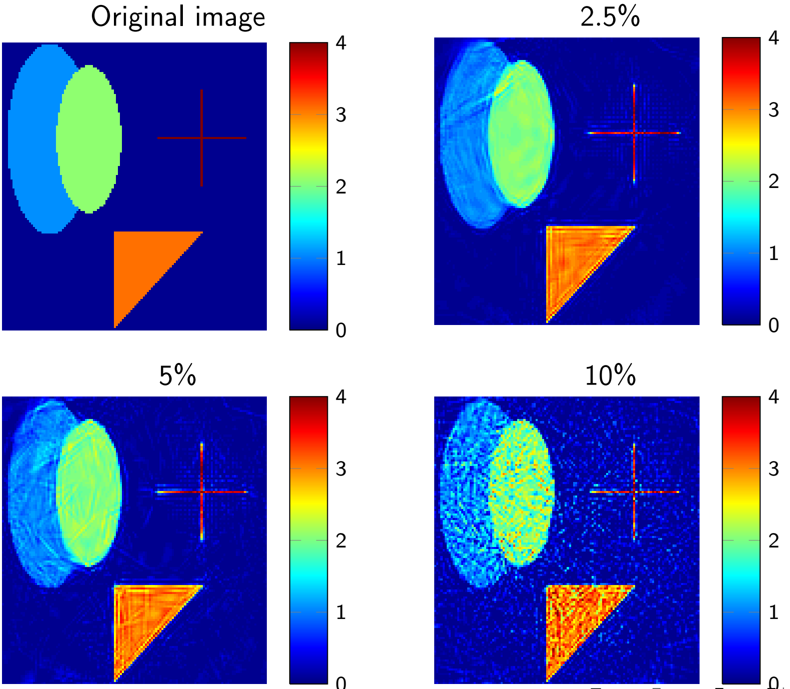
Original image



Reconstructed image ($\alpha = 10^{-4}$),



Solution has 1472 non zero coefficients (998460 shearlets).


Original image
2.5%
5%
10%
4
3
2
1
0

Current work

1. Parameter choice rule
   - Choice of α and stopping criterion
   - Different penalty for different scales?
2. Replacement of the Cartesian grid by a polar grid
   - Sparse FFT instead of NFFT
3. Penalizing the shearlet coefficients in the frequency domain

References I

Görner, Torsten, Ralf Hielscher, and Stefan Kunis (2012). "Efficient and accurate computation of spherical mean values at scattered center points." In: *Inverse Probl. Imaging* 6.4, pp. 645–661.

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Keiner, Jens, Stefan Kunis, and Daniel Potts. *NFFT 3.0, C subroutine library*. http://www.tu-chemnitz.de/~potts/nfft.